GRADED POSETS ZETA MATRIX FORMULA

Summary
The way to arrive at formula of zeta matrix for any graded posets with the finite set of minimal elements is delivered following [1]. This is being achieved via adjacency and zeta matrix description of bipartite digraphs chains – the representatives of graded posets. The bipartite digraphs elements of such chains amalgamate to form corresponding cover relation graded poset digraphs with corresponding adjacency matrices being amalgamated throughout natural join as special adequate database operation. The colligation of reachability and connectivity with the presented description is made explicit. The special posets encoded via KoDAGs directed acyclic graphs as cobeb posets Hasse diagrams are recognized as an example of differential posets subfamily. As on the 01.01.2009 one reminisce 261-th anniversary of death of Johann Bernoulli the First this Sylvester Night article is to commemorate this date.

1. Preliminaries: notation and terminology

1.1. We shall try to keep track of NIST Dictionary of Algorithms and Data Structures Terminology. Abbreviation: directed acyclic graph = DAG.

Note: The transitive closure of a directed acyclic graph or DAG is the reachability relation of the DAG and a strict partial order.

1.2. The following convention scheme is adopted: directed graph representatives of binary relations scheme are:

bipartite digraph representative \( D_R = (A \times A, R) \leftrightarrow R \subseteq A \times A \)

\( \equiv \) “just” digraph representative \( D(R) \equiv D_R = (A, E), E \leftrightarrow R \subseteq A \times A, \)

Dedicated to Professor Roman S. Ingarden
on the occasion of his ninetieth birthday
bipartite digraph representative $D_R = (A \times B, R) \iff R \subseteq A \times B$.

1.3. A directed path, is an oriented simple path with all arcs of the same direction i.e. all internal nodes have in- and out-degrees equal one.

Comment 1. “A directed path is a natural join of arcs that thus form a chain of vertices”, “A chain of coded data objects is a natural join of their subsequent pairs”.

Anticipated: coded data objects = relations (with varying arity allowed), binary relations, bipartite digraphs, adjacency matrices (of graphs or digraphs): [1, 2].

Comment 2. Because of immense number of applications of digraphs beyond mathematics – frequently successfully done by non-mathematicians – it happens sometimes that various names are given for the same notions and different names for the same objects. Let us recall and/or establish some of them.

Recall. Reachability is the ability reach some other vertex from a given vertex in a directed graph. For a directed graph $D = (\Phi, E), E \subseteq \Phi \times \Phi$ the reachability relation of $D$ is its transitive closure of $E$, i.e. the set of all ordered pairs $(s, t)$ of vertices in $\Phi$ for which there exist vertices $v_0 = s, v_1, ..., v_e = t$ such that $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq e$.

We define here (for directed graphs - more than nontrivial) the reachability = connectivity partial order relation $R$ over the nodes of the DAG as such that $xRy$ iff there exists a directed path from $x$ to $y$.

1.4. Relations set sum

reachability $\cup$ reflexivity = reflexive reachability,
reachability = connectivity,

This means that:

1.4.1. The reachability = connectivity relation is

$R^\infty = \bigcup_{k>0} R^k = \text{transitive closure of } R,$ i.e.,

$R^\infty = R^0 \cup R^1 \cup ... \cup R^n \cup ... \iff A(R^\infty) = A(R) \oplus A(R) \oplus ... \oplus A(R) \oplus ... ,$

where $A(R)$ is the Boolean adjacency matrix of the relation $R$ simple digraph and $\oplus$ stays for Boolean product.

The reflexive reachability relation $\zeta(R) \equiv R^*$ is defined as

$R^* = R^0 \cup R^1 \cup R^2 \cup ... \cup R^n \cup ... \bigcup_{k\geq0} R^k = R^\infty \cup I_A =$

$= \text{transitive and reflexive closure of } R \iff$

$\iff A(R^\infty) = A(R) \oplus A(R) \oplus A(R) \oplus ... \oplus A(R) \oplus ... .$

Comment 3. Colligate and identify $\zeta(R) \equiv R^*$ with incidence algebra zeta function and with zeta matrix of the poset associated to its Hasse digraph.
1.4.2. (Notation continued). In what follows we shall use mathematical terms: reachability and reflexive reachability according to: put $R = \prec \cdot$ which is Hasse diagram i.e. cover relation digraph - notation. Then note the schemes below.

The partial order $\leq$ for locally finite poset $\Pi = (\Phi, \leq)$ with respect to $\Pi$'s cover relation $\prec \cdot$ is:

1. $\prec$ the connectivity relation in Hasse digraph i.e. digraph $D = (\Phi, \prec \cdot)$;
2. $\leq$ the reflexive reachability relation in Hasse digraph i.e. digraph $D = (\Phi, \prec \cdot)$.

Schemes:

\[
\begin{align*}
\leq & = \prec \cdot \infty = \text{connectivity of } \prec \cdot \\
\leq & = \prec \cdot^* = \text{reflexive reachability of } \prec \cdot \\
\prec \cdot^* & = \zeta(\prec \cdot).
\end{align*}
\]

1.4.3. $\zeta(\prec \cdot)$ is also used to denote zeta matrix of the graded poset $P(D) = \Pi = (\Phi, \leq)$ associated to $D = (\Phi, \prec \cdot)$ [1,2] which is equivalent to say that $P(D) = (\Phi, \leq)$ = transitive, reflexive closure of $D = (\Phi, \prec \cdot)$.

1.5. In order to get complete graded digraph connect any two vertices lying on consecutive levels with an arc keeping one direction — say — upwards (see KoDAG in [1, 2]).

2. Natural join of adjacency matrices

2.1. Locally finite poset $\Pi$ if fixed, is denominated by all its covering pairs and vice versa of course i.e.

\[
\Pi = (\Phi, \leq) \leftrightarrow (\Phi, \prec \cdot),
\]

as all properties of order origin follow from those of transitivity requirement - and vice versa of course. Specifically recall-note the equivalence of descriptions:

The complete graded poset $\leftrightarrow$ The complete graded digraph.

The arcs of any digraph $G = (\Phi, E)$ with no multiple edges stay automatically for arcs of cover relation $\prec \cdot$ in the corresponding poset $\Pi = (\Phi, \leq)$ for which the digraph $G = (\Phi, E)$ becomes Hasse diagram i.e. we have:

\[
\leq = \prec \cdot^* = \text{reflexive reachability of } \prec \cdot \\
\prec \cdot^* & = (I - \prec \cdot)^{-1} \equiv \zeta(\prec \cdot).
\]

Except differently stated, we shall identify a digraph with its adjacency matrix. In our context these are to be Hasse digraphs $D = (\Phi, \prec \cdot)$ of graded posets $\Pi = (\Phi, \leq)$.

There are three standard widely used prevalent encodings, three ways of portraying partially ordered sets $P(D) = (\Phi, \leq)$: Hasse diagrams $D = (\Phi, \prec \cdot)$; zeta matrices $\zeta(\leq)$; and cover matrices $\zeta(\prec \cdot)$. These matrices are of course the adjacency matrices of the corresponding digraphs $(\Phi, \leq)$ and $(\Phi, \prec \cdot)$. In the incidence algebra description of locally finite posets $\zeta(\leq)$ may be identified with the incidence function.
i.e., the characteristic function of the partial order \( \leq \), (see [1] and references therein for the source cobweb posets examples of these objects). Here down we shall use adjacency and biadjacency nomenclature [1, 2].

**Examples of** \( \zeta(\leq) \)

Let \( F \) denotes arbitrary natural numbers valued sequence. Let \( A_N \) be the Hasse matrix i.e. adjacency matrix of cover relation \( \prec \cdot \) digraph denominated by sequence \( N [1] \). Then the zeta matrix \( \zeta = (1 - A_N)^{-1} \) for the denominated by \( F = N \) cobweb poset is of the form [1].

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

Fig. \( \zeta_N \): The incidence matrix \( \zeta \) for the natural numbers i.e. \( N \)-cobweb poset.

Note that the matrix \( \zeta \) representing uniquely its corresponding cobweb poset does exhibits a staircase structure of zeros above the diagonal (see above, see below) which is characteristic to and characteristics of Hasse diagrams of all cobweb posets while for graded posets it is characteristic too – this time all together with additional zeros right to the staircase – zeros generated by the biadjacency matrices \( \langle B_k \rangle_{k \geq 0} \) chain as described in Observation 4. Another words: it is the natural join \( \oplus \to \) chain

\[
\oplus \to \bigoplus_{k=0}^{n} B_k, \ n \in N \cup \{ \infty \}
\]

which is equivalent to \( \zeta \) function characteristics of any fixed \( F \)-denominated graded poset \( (P, \leq) \).
Fig. $\zeta_F$: The matrix $\zeta$ for the Fibonacci cobweb poset associated to $F$-KoDAG Hasse digraph.

Comment 4. The given $F$-denominated staircase zeros structure above the diagonal of zeta matrix $\zeta$ is the unique characteristics of its corresponding $F$-KoDAG Hasse digraphs.

2.2. The natural join condition

The natural join operation is a binary operation like $\Theta$ operator in computer science denoted here by $\oplus\to$ symbol deliberately referring – in a quite reminiscent manner – to direct sum $\oplus$ of adjacency Boolean matrices and – as matter of fact and in effect – to direct the sum $\oplus$ of corresponding biadjacency [reduced] matrices of digraphs under natural join.

$\oplus\to$ is a natural operator for sequences construction. $\oplus\to$ operates on multi-ary relations according to the scheme:

$$(n + k)_{\text{ary}} \oplus\to (k + m)_{\text{ary}} = (n + k + m)_{\text{ary}},$$

For example:

$$(1 + 1)_{\text{ary}} \oplus\to (1 + 1)_{\text{ary}} = (1 + 1 + 1)_{\text{ary}},$$

binary $\oplus\to$ binary = ternary.

Accordingly an action of $\oplus\to$ on these multi-ary relations’ digraphs adjacency matrices is to be designed soon in what follows.
Domain-Codomain $F$-sequence condition [1]

\[ \text{dom}(R_{k+1}) = \text{ran}(R_k), \quad k = 0, 1, 2, \ldots . \]

Consider any natural number valued sequence $F = \{F_n\}_{n \geq 0}$. Consider then any chain of binary relations defined on pairwise disjoint finite sets with cardinalities appointed by $F$-sequence elements values. For that to start we specify at first a relations’ domain-co-domain $F$-sequence.

**Domain-Codomain $F$-sequence**

\[ (|\Phi_n| = F_n) \]

Let $\Phi = \bigcup_{k=0}^{\infty} \Phi_k$ be the corresponding ordered partition [ anticipating - $\Phi$ is the vertex set of $D = (\Phi, \prec \cdot)$ and its transitive, reflexive closure $(\Phi, \leq)$]. Impose \( \text{dom}(R_{k+1}) = \text{ran}(R_k) \) condition, $k \in N \cup \{\infty\}$. What we get is binary relations chain.

**Definition 1.** (Relation’s chain) Let $\Phi = \bigcup_{k=0}^{\infty} \Phi_k$, $\Phi_k \cap \Phi_n = \emptyset$ for $k \neq n$, $|\Phi_n| = F_n$; $i, k, n = 0, 1, 2, \ldots$. Then the sequence $\langle R_k \rangle_{k \geq 0}$ is called natural join (binary) relation’s chain.

Extension to varying arity relations’ natural join chains is straightforward.

As necessarily $\text{dom}(R_{k+1}) = \text{ran}(R_k)$ for relations’ natural join chain any given binary relation’s chain is not just a sequence therefore we use ”link to link” notation for $k, i, n = 1, 2, 3, \ldots$ ready for relational data basis applications:

\[ R_0 \oplus \rightarrow R_1 \oplus \rightarrow ... \oplus \rightarrow R_i \oplus \rightarrow ... \oplus \rightarrow R_i + n, \ldots \]

where $\oplus \rightarrow$ denotes natural join of relations as well as both natural join of their bipartite diognets and the natural join of their representative adjacency matrices (see [1, 2]).

Relation’s $F$-chain naturally represented by [identified with] the chain of theirs bipartite diognets

\[ R_0 \oplus \rightarrow R_1 \oplus \rightarrow ... \oplus \rightarrow R_i \oplus \rightarrow ... \oplus \rightarrow R_i + n, \ldots \Leftrightarrow \]

\[ \Leftrightarrow B_0 \oplus \rightarrow B_1 \oplus \rightarrow ... \oplus \rightarrow B_i \oplus \rightarrow ... \oplus \rightarrow B_i + n, \ldots \]

results in $F$-partial ordered set $\langle \Phi, \leq \rangle$ with its Hasse digraph representation looking like specific “cobweb” image (for cobweb posets portraits see [1] and references therein and see also cobwebs in action on http://www.faces-of-nature.art.pl/cobwebposets.html).

2.3. Partial order $\leq$. The partial order relation $\leq$ in the set of all points-vertices is determined uniquely by the above equivalent $F$-chains. Let $x, y \in \Phi = \bigcup_{k=0}^{\infty} \Phi_k$ and let $k, i = 0, 1, 2, \ldots$. Then

\[ x \leq y \Leftrightarrow \forall x \in \Phi : x \leq x \lor \Phi_1 \exists x < y \in \Phi_{i+k} \iff x(R_i \odot \ldots \odot R_{i+k-1})y \]
where “⊙” stays for [Boolean] composition of binary relations.

Relation \((\leq)\) is defined equivalently: \(x \leq y\) in \((\Phi, \leq)\) iff either \(x = y\) or there exist a directed path from \(x\) to \(y\); \(x, y \in \Phi\).

Let now \(R_k = \Phi_k \times \Phi_{k+1}, k \in \mathbb{N} \cup \{0\}\). For “historical” reasons \([1]\) we shall call such partial ordered set \(\Pi = (\Phi, \leq)\) the **cobweb poset** as theirs Hasse digraph representation looks like specific “cobweb” image.

2.4. The natural join \(\oplus \rightarrow\) operation and the natural join of matrices satisfying the natural join condition. We define here the adjacency matrices representation of the natural join \(\oplus \rightarrow\) operation.

The adjacency matrix \(A\) of a bipartite graph with biadjacency = reduced adjacency matrix \(B\) is given by

\[
A = \begin{pmatrix}
0 & B \\
B^T & 0
\end{pmatrix}.
\]

**Definition 2.** The adjacency matrix \(A[D]\) of a bipartite digraph \(D(R) = (P \cup L, E \subseteq P \times L)\) with biadjacency matrix \(B\) is given by \([1]\)

\[
A[D] = \begin{pmatrix}
0_{k,k} & B(k \times m) \\
0_{m,k} & 0_{m,m}
\end{pmatrix}.
\]

where \(k = |P|, m = |L|\).

**Note:** biadjacency and cover relation \(\prec\) matrix for bipartite digraphs coincide. By extension - we shall call cover relation \(\prec\) matrix also the biadjacency matrix.

**Convention 1.** \(S \odot R = \text{composition of binary relations} S \text{ and } R \leftrightarrow B_{R \odot S} = B_R \odot B_S\) where \((|V| = k, |W| = m)\); \(B_R(k \times m) \equiv B_R\).

\(B_R\) is the \((k \times m)\) biadjacency [or another name: reduced adjacency] matrix of the bipartite relations’ \(R\) digraph \(B(R)\) and \(\odot\) apart from relations composition denotes also Boolean multiplication of these rectangular biadjacency Boolean matrices \(B_R, B_S\). What is their form? The answer is in the block structure of the standard square \((n \times n)\) adjacency matrix \(A[D(R)]; n = k + m\). The form of standard square adjacency matrix \(A[G(R)]\) of bipartite digraph \(D(R)\) has the following apparently recognizable block reduced structure: \([O_{s \times s}\) stays for \((k \times m)\) zero matrix\]

\[
A[D(R)] = \begin{pmatrix}
O_{k \times k} & A_R(k \times m) \\
A_{m \times k} & O_{m \times m}
\end{pmatrix}.
\]

Let \(D(S) = (W(S) \cup T(S), E(S)); W \cap T = \emptyset, E(S) \subseteq W \times T; (|W| = m, |T| = s)\); hence

\[
A[D(S)] = \begin{pmatrix}
O_{m \times m} & A_S(m \times s) \\
O_{s \times m} & O_{s \times s}
\end{pmatrix}.
\]
Definition 3 (natural join condition). The ordered pair of matrices \((A_1, A_2)\) is said to satisfy the natural join condition iff they have the block structure of \(A[D(R)]\) and \(A[D(S)]\) as above i.e. iff they might be identified accordingly: \(A_1 = A[D(R)]\) and \(A_2 = A[D(S)]\).

Correspondingly if two given digraphs \(G_1\) and \(G_2\) are such that their adjacency matrices \(A_1 = A[G_1]\) and \(A_2 = A[G_2]\) do satisfy the natural join condition we shall say that \(G_1\) and \(G_2\) satisfy the natural join condition. For matrices satisfying the natural join condition one may define what follows.

First we define the Boolean reduced or natural join composition \(\odot\rightarrow\) and secondly the natural join \(\oplus\rightarrow\) of adjacent matrices satisfying the natural join condition.

Definition 4 (\(\odot\rightarrow\) composition).

\[
A[D(R \odot S)] =: A[D(R)] \odot A[D(S)] = \begin{bmatrix}
O_{k \times k} & A_{R \odot S}(k \times s) \\
O_{s \times k} & O_{s \times s}
\end{bmatrix}
\]

where \(A_{R \odot S}(k \times s) = A_R(k \times m) \odot A_S(m \times s)\). According to the scheme:

\[
[(k + m) \times (k + m)] \odot \rightarrow [(m + s) \times (m + s)] = [(k + s) \times (k + s)].
\]

Comment 5. The adequate projection makes out the intermediate, joint in common \(\text{dom}(S) = \text{rang}(R) = W\), \(|W| = m\).

The above Boolean reduced composition \(\odot\rightarrow\) of adjacent matrices technically reduces then to the calculation of just Boolean product of the reduced rectangular adjacency matrices of the bipartite relations’ graphs.

We are however now in need of the Boolean natural join product \(\oplus\rightarrow\) of adjacent matrices already announced at the beginning of this presentation. Let us now define it.

As for the natural join notion we aim at the morphism correspondence:

\[
S \oplus\rightarrow R \iff M_{S \oplus\rightarrow R} = M_R \oplus\rightarrow M_S
\]

where \(S \oplus\rightarrow R = \text{natural join of binary relations } S \text{ and } R\) while \(M_{S \oplus\rightarrow R} = M_R \oplus\rightarrow M_S = \text{natural join of standard square adjacency matrices (with customary convention: } M[G(R)] \equiv M_R \text{ adapted)}\). Attention: recall here that the natural join of the above binary relations \(R \oplus\rightarrow S\) is the ternary relation – and thus one results in \(k\)-ary relations if with more factors undergo the \(\oplus\rightarrow\) product. As a matter of fact \(\oplus\rightarrow\) operates on multi-ary relations according to the scheme:

\[(n + k)_{ary} \oplus\rightarrow (k + m)_{ary} = (n + k + m)_{ary}.
\]

For example: \((1 + 1)_{ary} \oplus\rightarrow (1 + 1)_{ary} = (1 + 1 + 1)_{ary}, \text{ binary} \oplus\rightarrow \text{binary} = \text{ternary}.

Technically – the natural join of the \(k\)-ary and \(n\)-ary relations is defined accordingly the same way via \(\oplus\rightarrow\) natural join product of adjacency matrices – the adjacency matrices of these relations’ Hasse digraphs.
With the notation established above we finally define the natural join $\oplus \rightarrow$ of two adjacency matrices as follows:

**Definition 5** [natural join $\oplus \rightarrow$ of matrices].

$$A[D(R) \oplus \rightarrow S] =: A[D(R)] \oplus \rightarrow A[D(S)] =$$

$$= \begin{bmatrix} O_{k \times k} & A_R(k \times m) \\ O_{m \times k} & O_{m \times m} \end{bmatrix} \oplus \rightarrow \begin{bmatrix} O_{m \times m} & A_S(m \times s) \\ O_{s \times m} & O_{s \times s} \end{bmatrix} =$$

$$= \begin{bmatrix} O_{k \times k} & A_R(k \times m) & O_{k \times s} \\ O_{m \times k} & O_{m \times m} & A_S(m \times s) \\ O_{s \times k} & O_{s \times m} & O_{s \times s} \end{bmatrix}.$$

**Comment 6.** The adequate projection used in natural join operation lefts one copy of the joint in common “intermediate” submatrix $O_{m \times m}$ and consequently lefts one copy of “intermediate” joint in common $m$ according to the scheme:

$$[(k + m) \times (k + m)] \oplus \rightarrow [(m + s) \times (m + s)] \equiv [(k + m + s) \times (k + m + s)].$$

### 2.5. The biadjacency i.e cover relation $\prec \cdot$ matrices of the natural join of adjacency matrices.

Denote with $B(A)$ the biadjacency i.e cover relation $\prec \cdot$ matrix of the adjacency matrix $A$.

**Note:** biadjacency and cover relation $\prec \cdot$ matrix for bipartite digraphs coincide.

By extension – we shall call cover relation $\prec \cdot$ matrix the biadjacency matrix too, as for any graded digraph with more than one level we might represent it as a partition into two independent sets, though it is more natural to see it as the natural join of the sequence of bipartite digraphs.

Let $A(G)$ denotes the adjacency matrix of the digraph $G$, for example a di-biclique relation digraph. Let $A(G_k), k = 0, 1, 2, \ldots$ be the sequence adjacency matrices of the sequence $G_k, k = 0, 1, 2, \ldots$ of digraphs. Let us identify $B(A) \equiv B(G)$ as a convention.

**Definition 6** [digraphs natural join]. Let digraphs $G_1$ and $G_2$ satisfy the natural join condition. Let us make then the identification $A(G_1 \oplus \rightarrow G_2) \equiv A_1 \oplus \rightarrow A_2$ as definition. The digraph $G_1 \oplus \rightarrow G_2$ is called the digraphs natural join of digraphs $G_1$ and $G_2$. Note that the order is essential.

We observe at once what follows.

**Observation 1.**

$$B(G_1 \oplus \rightarrow G_2) \equiv B(A_1 \oplus \rightarrow A_2) = B(A_1) \oplus B(A_2) \equiv B(G_1) \oplus B(G_2).$$
Comment 7. The Observation 1 justifies the notation $\oplus \rightarrow$ for the natural join of relations digraphs and equivalently for the natural join of their adjacency matrices and equivalently for the natural join of relations that these are faithful representatives of. Recall: $B(A)$ is the biadjacency i.e cover relation $\prec \cdot$ matrix of the adjacency matrix $A$.

Note: biadjacency and cover relation $\prec \cdot$ matrix for bipartite digraphs coincide. Recall that by extension – we call cover relation $\prec \cdot$ matrix the biadjacency matrix too.

As a consequence we have.

Observation 2.

\[
B (\oplus \rightarrow \bigoplus_{i=1}^{n} G_i) \equiv B[\oplus \rightarrow \bigoplus_{i=1}^{n} A(G_i)] = \bigoplus_{i=1}^{n} B[A(G_i)] \equiv \text{diag}(B_1, B_2, \ldots, B_n) = \\
\begin{bmatrix}
B_1 & & \\
& B_2 & \\
& & B_3 \\
& & & \ddots \\
& & & & \ddots \\
& & & & & B_n
\end{bmatrix},
\]

or equivalently

\[
\kappa \equiv \chi (\prec \cdot) = \chi (\oplus \rightarrow \bigoplus_{i=1}^{n} \prec \cdot) \equiv \\
\begin{bmatrix}
0 & B_1 & & \\
0 & 0 & B_2 & \\
& & & \ddots \\
& & & & \ddots \\
& & & & & 0 & B_n
\end{bmatrix},
\]

\(n \in \mathbb{N} \cup \{\infty\}\)

2.6. The formula of zeta matrix for graded posets with the finite set of minimal elements i.e for $F$-graded posets. Any graded poset with the finite set of minimal elements is a $F$-sequence denominated sub-poset of its corresponding cobweb poset.

The Observation 2 supplies the simple recipe for the biadjacency (reduced adjacency) matrix of Hasse digraph coding any given graded poset with the finite set of minimal elements. The recipe for zeta matrix is then standard. We illustrate this by the source example; the source example as the adjacency matrices i.e. zeta matrices of any given graded poset with the finite set of minimal elements are sub-matrices of their corresponding cobweb posets and as such have the same block matrix structure.

The explicit expression for zeta matrix $\zeta_F$ of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued $F$-sequence was given in [1] due to more than mnemonic efficiency of the up-side-down notation being applied
(see [1] and references therein). With this notation inspired by Gauss and replacing
\( k \)-natural numbers with “\( kF \)” numbers one gets

\[
A_F = \begin{bmatrix}
0_{1F \times 1F} & I(1F \times 2F) & 0_{1F \times \infty} \\
0_{2F \times 1F} & 0_{2F \times 2F} & I(2F \times 3F) & 0_{2F \times \infty} \\
0_{3F \times 1F} & 0_{3F \times 2F} & 0_{3F \times 3F} & I(3F \times 4F) & 0_{3F \times \infty} \\
0_{4F \times 1F} & 0_{4F \times 2F} & 0_{4F \times 3F} & 0_{4F \times 4F} & I(4F \times 5F) & 0_{4F \times \infty}
\end{bmatrix}
\]

and

\[
\zeta_F = \exp[\oplus] [A_F] \equiv (1 - A_F)^{-1} \oplus \equiv I_{\infty \times \infty} + A_F + A_F^{\oplus^2} + ... =
\]

\[
\begin{bmatrix}
I_{1F \times 1F} & I(1F \times \infty) \\
0_{2F \times 1F} & I_{2F \times 2F} & I(2F \times \infty) \\
0_{3F \times 1F} & 0_{3F \times 2F} & I_{3F \times 3F} & I(3F \times \infty) \\
0_{4F \times 1F} & 0_{4F \times 2F} & 0_{4F \times 3F} & I_{4F \times 4F} & I(4F \times \infty)
\end{bmatrix}
\]

where \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k\) and \( n \in N \cup \{\infty\}\).

**Observation 3.** Let us denote by \( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) (see the authors papers quoted) the di-bicliques denominated by subsequent levels \( \Phi_k, \Phi_{k+1} \) of the graded \( F \)-poset \( P(D) = (\Phi, \preceq) \) i.e. levels \( \Phi_k, \Phi_{k+1} \) of its cover relation graded digraph \( D = (\Phi, \preceq) \) [Hasse diagram]. Then

\[
B_{\oplus \rightarrow^{n}_{k=1} \langle \Phi_k \rightarrow \Phi_{k+1} \rangle} = \text{diag}(I_1, I_2, ..., I_n) =
\]

\[
\begin{bmatrix}
I(1F \times 2F) \\
I(2F \times 3F) \\
I(3F \times 4F) \\
... \\
I(nF \times (n+1)F)
\end{bmatrix}
\]

where \( I_k \equiv I(kF \times (k+1)F) \), \( k = 1, \ldots, n \) and where – recall – \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k\) and \( n \in N \cup \{\infty\}\).

The recipe for any \( F \)-denominated i.e. the recipe for any graded poset with a
finite minimal elements set is supplied via the following observation.

**Observation 4.** Consider bigraphs’ chain obtained from the above di-bicliques’ chain
via deleting or no arcs making thus [if deleting arcs] some or all of the di-bicliques
\( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) not di-bicliques; denote them as \( G_k \). Let \( B_k = B(G_k) \) denotes their bi-
adjacency matrices correspondingly. Then for any such \( F \)-denominated chain [hence
any chain] of bipartite digraphs \( G_k \) the general formula is:
\[ B(\oplus_{i=1}^n G_i) \equiv B(\oplus_{i=1}^n A(G_i)) = \oplus_{i=1}^n B[A(G_i)] \equiv \text{diag}(B_1, B_2, \ldots, B_n) = \\
\begin{bmatrix}
B_1 & B_2 & \cdots & B_n \\
\end{bmatrix} \\
n \in N \cup \{\infty\}.
\]

Comment 8. Note the notation identification: \( \zeta_F = \exp[\mathbf{A}_F] \equiv (1 - \mathbf{A}_F)^{-1} \).

Note that \( n! = 1 \mod 2 \). Colligate also aside (?) reason:

\[ \lim_{q \to 1} \exp_q = \exp \] 

while

\[ \lim_{q \to 0} \exp_q[x] = (1 - x)^{-1}. \]

Consult the Remark in [1] on the cases: Boolean poset \( 2^N \) and the “Ferrand-Zecken-dorf” poset of finite subsets of \( N \) without two consecutive elements.

Observation 5. The \( F \)-poset \( P(G) = (\Phi, \leq) \) or equivalent to say: its cover relation graded digraph \( G = (\Phi, \prec \cdot) = \oplus_{k=0}^m G_k \) is of Ferrers dimension one iff in the process of deleting arcs from the cobweb poset Hasse diagram \( D = (\Phi, \prec \cdot) = \oplus_{k=0}^n \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) does not produces \( 2 \times 2 \) permutation submatrices in any of bigraphs \( G_k \) biadjacency matrices \( B_k = B(G_k), k = 0, \ldots, n, n \in N \cup \{\infty\} \).

3. Cobweb posets and differential posets of Stanley [3, 4]

3.1. Preliminaries

The graded digraph \( G \) (graph) is utterly denominated by its sequence of bipartite digraphs (graphs) that every two consecutive levels of \( G \) do constitute.

The complete graded digraph \( D \) is utterly denominated by its sequence of complete bipartite digraphs – di-bicliques [1] that every two consecutive levels of \( D \) do constitute (see KoDAGs in [1] and references therein, consult the Example 2.7.2 in [5]). Because of their appearance, an at a first glance outlook – these complete graded digraph \( D \) associated posets where called cobweb posets [digraphs \( D \) are identified with Hasse diagram of cobweb posets (see KoDAGs in [1, 2] and references therein)].

Comment 9. The appearance of “almost complete” graded digraph (subgraphs of KoDAGs) is tremendously prevailing. These look like – the hoary tree with silver cobweb threads. The other extreme to the complete in such a picture of a tree with cobweb (KoDAG) is an also beautiful melancholic bare rooted directed tree graph – void of this spider’s web hoary tunicate and droplets (loops). Here are come some examples listed with the convention that graphs become digraphs with all arcs
directed upward (in the direction of increasing rank) or dually – downwards (in the
direction of decreasing rank). If so has been done these become graded DAGs.

First let us establish-recall for clarity that the directed tree (all arcs directed
away from its root) is a digraph which becomes a tree if directions on the edges are
ignored. Colligate with an arborescence. Naturally every arborescence is a directed
acyclic graph.

Now come examples of sub-cobweb posets digraphs (hence DAGs) [5, 3, 4].
0. The binary tree digraph (directed Tree).
1. The Fibonacci digraph ([1], directed Tree).
2. The Young graph.
3. The Young-Fibonacci graph.
4. The Young-Fibonacci insertion graph.
5. The 2-dimensional Pascal graph.
6. The lattice of binary trees graph.
7. The lattice of Bracket tree graph.
8. The Fan graph [5] (Amalgamate $k$ disjoint infinite chains by gluing their roots
   ("zeros").)
9. The special complete graded graph [5] (the set of vertices the same as in 8, connect
   any two vertices from consecutive levels $\Phi_k$ and $\Phi_{k+1}$ with an upward directed arc).
10. The $F$-denominated (hence any) complete graded digraph with finite minimal
    elements set (KoDAG in [1, 2] and references therein).

3.2. Cobweb posets and differential posets

The class of posets known as differential posets were first introduced and studied by
raising and lowering operators $U$ and $D$ which satisfy the commutation relation
$DU - UD = rI$ for some integer $r > 0$. Generalizations of this class of posets
were studied by Stanley [4] and Fomin [5]. A number of examples of generalized
differential posets are given in these [3, 4, 5] papers. Another example, a poset of
rooted unlabelled trees, was introduced by Hoffman [6].

Let us consider at first the case $r = 1$ which we shall call GHW case for the
reasons to become apparent soon.

In this GHW $r = 1$ case the operators $U$ and $D$ are defined correspondingly,
$(x, y, z \in \Phi)$:

**Definition 7.**

\[
\begin{align*}
Dx &= \sum_{y \prec x} y, \\
Ux &= \sum_{x \prec y} y
\end{align*}
\]

extended by linearity to (say complex, or...) linear space $C[\Phi]$.

From Theorem 2.2. in [3] we then have (consult also [7]) that GHW commutation
relation $DU - UD = I$ holds iff $P(D) = (\Phi, \leq)$ is differential ($r = 1$) poset. Out of
this one infers inductively [7–11] what follows.
Observation 6.  

\[ DU^n = nU^{n-1} + U^n D, \]

for \( n \in N \).

Observation 7. Cobweb posets Hasse digraphs from 10. above are examples of \( \overrightarrow{q}, \overrightarrow{r} \) – differential posets (see [5] for all \( q_n = 1 \)) and might serve for more general structures (due to Fomin [5]) called Dual graded Graphs.

Check. Indeed. In [5] \( U_n \) and \( D_n \) are defined as restrictions of \( U \) and \( D \) onto \( C[\Phi_k] \) “homogeneous” subspaces of \( C[\Phi] \), \( k = 0, 1, 2, \ldots \). Then

\[ D_{n+1} = q_n U_{n-1} D_n + r_n I_n, \quad n \in N, \]

where \( r_0 = 1_F, q_0 = 0 \) for \( n = 0 \) and

\[ q_n = \frac{(n+1)}{(n-1)}F, \quad r_n = 0 \quad \text{for} \quad n > 0. \]

For Fomin examples 8. and 9. above (see 2.7.2. in [5]) the \( (1.4.11) \) i.e. the \( (1.4.10) \) with all \( q_n = 1 \) from [5] holds for \( r_0 = 0, r_1 = r_2 = r_3 \ldots = 0. \)

The KoDAGs graded digraphs example 10 in view of Observation 7 and Observation 6 becomes the motivating example of the following description appealing to the corresponding R. Stanley definition from [3]. One expect this description to be efficient and tangible while the combinatorics of path counting is concerned also via colligation with quantum models – for example.

Definition 8. Let \( F \) be such that \( 0_F = 1 \). The locally finite graded poset \( \Pi = \langle \Phi, \leq \rangle \) is then said to be \( F \)-differential poset iff

1. if \( x, y \in \Phi, x \neq y \) and there are \( k_F \) elements of \( \Phi \) which cover both of them then there are exactly \( k_F \) elements of \( \Phi \) which are covered by both of them;
2. if \( x \in \Phi \) covers exactly \( k_F \) elements from \( \Phi \) then this very \( x \) is covered exactly by \((k + 1)_F \) elements from \( \Phi \).

As in \( F = N \) case (i.e for all \( k = 1, 2, \ldots \) we have \( k_F \equiv k_N \equiv k \)) the above conditions determine the number of elements and cover relations among them in the up to fourth rank [7]. Above the fourth rank \((k > 4)\) cover relations and number of elements in levels \( \Phi_k \) vary. And what does happen when the requirement \( 0_F = 1 \) is relaxed?

Let us introduce the direct sum of certain projection on \( \Phi_n \) operators and denote this operator with the symbol \( \delta_F \). Let \( x_n \) denotes homogeneous element of \( \Phi \) i.e \( x \in \phi_n \). Then \( \delta_F \) is specified as follows.

Definition 9. 

\[ \delta_F = \text{diag}(1_F-0_F, 2_F-1_F, 3_F-2_F, \ldots, (n+1)_F-n_F, \ldots) \equiv \text{diag}(\delta_0, \delta_1, \delta_2, \ldots, \delta_n, \ldots) \]

i.e. \( \delta_F(x_n) = \delta_n x_n \).
A straightforward verifying (see [7]) leads us to thus confirmed conclusion below; (note that $\delta_N = I$).

**Observation 8.**

$$DU - UD = \delta_F.$$  

Out of this (as in [10, 11]) one infers inductively what follows.

**Observation 9.**

$$DU^n = n\delta_F U^{n-1} + U^n D,$$

for $n \in N$.

Since $Dx_0 = 0$ we have $DU = n\delta_F U^{n-1}x_0$ which means that $D$ is representative of the Markowsky general linear operator i.e., a derivative from extended umbral calculus (see [10, 11, 12] and plenty of references therein). The $F$-denominated cobweb posets are in a sense a canonical example of $F$-differential posets. More on that is expected soon.

**Comment 10** (miscellaneous (aside?) final remark).

The ingenious ideas of differential and dual graded posets that we owe to Stanley [3, 4] and Fomin [5] bring together combinatorics, representation theory, topology, geometry and many more specific branches of mathematics and mathematical physics thanks to intrinsic ingredient of these mathematical descriptions which is the Graves-Heisenberg-Weyl (GHW) algebra usually attributed to Heisenberg by physicists and to Herman Weyl by mathematicians and sometimes to both of them (see: [3] for Weyl, [5] for Heisenberg and then [8] and [9]; for GHW see [10–12] then note the content and context of [13, 14] ). As noticed by the author of [9] the formula

$$[f(a), b] = cf'(a)$$

where

$$[a, b] = c, [a, c] = [b, c] = 0$$

pertains to Charles Graves from Dublin [8]. Then it was re-discovered by Paul Dirac and others in the next century.

Let us then note that the picture that emerges in [11, 12] discloses the fact that any umbral representation of finite (extended) operator calculus or equivalently - any umbral representation of GHW algebra makes up an example of the algebraization of the analysis with generalized differential operators acting on the algebra of polynomials or other algebras as for example formal series algebras.

**Bibliography remark.** On Umbra Difference Calculus references streams see [15] including references and Comment 8 and all of that. On the history of cobweb poset $\zeta$ function formulas and also for the $\zeta_{-1}$ formula see [16].
References


FORMUŁA NA MACIERZ ZETA
CZĘŚCIOWO UPORZĄDKOWANEGO ZBIORU Z GRADACJĄ

STRESZCZENIE

Podano jawnie i udowodniono formułę na macierz zeta dowolnego częściowo uporządkowanego zbioru ("poset") ze stopniowaniem (gradacją) o skończonej liczbie elementów minimalnych.

Cel ten jest osiągnięty dzięki owych "cobweb posets" jako i dowolnych (posets) częściowo uporządkowanych zbiorów ze stopniowaniem (gradacją) o skończonej liczbie elementów minimalnych utożsamieniu ze złączeniem naturalnym (natural join) łańcuchów grafów dwudzielnych.

Odwzorowuje to skutkująco strukturę macierzy sąsiedztwa wszystkich acyklicznych grafów skierowanych zwanych diagramami Hasse tych "posetów" z gradacją. Jest to mianowicie postać sekwencyjnego złączenia naturalnego macierzy składowych łańcuchów grafów dwudzielnych.

W przypadku szczególnych częściowo uporządkowanych zbiorów ze stopniowaniem zwanych "cobweb posets" stanowiących w złączeniu naturalnym ciągi Kompletnych Grafów dwudzielnych – uporządkowanych (ordered) oraz skierowanych i acyklicznych (DAG’s) autor na część Profesora Kazimierza Kuratowskiego nazwał owe grafy Hasse’go – KoDAGs.

Zidentyfikowano owe KoDAGs jako szczególnie wyróżnione przykłady z rodziny częściowo uporządkowanych zbiorów z różniczkowaniem ("differential posets subfamily").