Summary

We arrive at the explicit formula for the inverse of zeta matrix for any graded posets with the finite set of minimal elements following the first reference which is referred to as SNACK that is Sylvester Night Article on Cobweb posets and KoDAG graded digraphs. In SNACK the way to arrive at formula of the zeta matrix for any graded posets with the finite set of minimal elements was delivered and explicit form was given. We present here effective way toward the formula for the inverse of zeta matrix which is being unearthed via adjacency and zeta matrix description of bipartite digraphs chains, the representatives of graded posets with sine qua non essential use of digraphs and matrices natural join introduced by the present author.

Namely, the bipartite digraphs elements of such chains amalgamate so as to form corresponding cover relation graded poset digraphs with corresponding adjacency matrices being amalgamated throughout natural join constituting adequate special database operation.

As a consequence apart from zeta function also the Möbius function explicit expression for any graded posets with the finite set of minimal elements is being arrived at.

Purposely, on the way – special number theoretic code-triangles for KoDAGs are proposed and apart from the author combinatorial interpretation of $F$-nomial coefficients another related interpretation is inferred while referring to the number of all maximal chains in the corresponding poset interval. The formula for August Ferdinand Möbius matrix is also interpreted combinatorially.

1. The upside down notation principle

We shall here take for granted the notation and the results of [1] which is referred to as SNACK that is Sylvester Night Article on Cobweb posets and KoDAG graded digraphs.
In particular, \(\langle \Pi, \leq \rangle\) denotes cobweb partial order set (cobweb poset) while \(I(\Pi, R)\) denotes its incidence algebra over the ring \(R\). Correspondingly, \(\langle P, \leq \rangle\) denotes arbitrary graded poset while \(I(P, R)\) denotes its incidence algebra over the ring \(R\).

For example, \(R\) might be taken to be Boolean algebra \(2^{\{1\}}\), the field \(Z_2 = \{0, 1\}\), the ring of integers \(Z\) or real or complex or \(p\)-adic fields. The present article is the next one in a series of papers listed in reversed order of appearence and these are: [1–3]. The authors upside down notation is used throughout this paper i.e. \(F_n \equiv n_F\). The Upside Down Notation Principle used since last century effectively (see [1–20] and for earlier references therein; in particular see Appendix in [9] copied from [32]) may be formulated as a Principle i.e. trivial, powerful statement as follows. Through all the paper \(F\) denotes a natural numbers valued sequence sometimes specified to be Fibonacci or others – if needed. Among many consequences of this is that graded posets (= their cover relation digraphs \(iff\) Hasse diagrams) are connected and sets of their minimal elements are finite.

**Comment 0.** Mantra. If the statement \(s(F)\) depends (relies, is based on, “lies in ambush”...) only on the fact that \(F\) is a natural valued numbers sequence then if the statement \(s(F)\) is proved true for \(F = N\) then it is true for any natural valued numbers sequence \(F\).

**The upside down notation principle**

1. Let the statement \(s(F)\) depends only on the fact that \(F\) is a natural numbers valued sequence.

2. Then if one proves that \(s(N) \equiv s(n)_{n \in N}\) is true – the statement 
   
   \[s(F) \equiv s(n_F)_{n \in N}\]

   is also true.

   Formally – use equivalence relation classes induced by co-images of \(s : \{F\} \mapsto 2^{\{1\}}\) and proceed in a standard way.

1.1. Ponderables

**Definition 1.** Let \(n \in N \cup \{0\} \cup \{\infty\}\). Let \(r, s \in N \cup \{0\}\). Let \(\Pi_n\) be the graded partial ordered set (poset) i.e.

\[\Pi_n = (\Phi_n, \leq) = \left( \bigcup_{k=0}^{n} \Phi_k, \leq \right) \text{ and } \langle \Phi_k \rangle_{k=0}^{n}\]

constitutes ordered partition of \(\Pi_n\). A graded poset \(\Pi_n\) with finite set of minimal elements is called cobweb poset iff

\[\forall x, y \in \Phi \text{ i.e. } x \in \Phi_r \text{ and } y \in \Phi_s : r \neq s \Rightarrow x \leq y \text{ or } y \leq x, \quad \Pi_\infty \equiv \Pi.\]

**Note.** By definition of \(\Pi\) being graded its levels \(\Phi_r \in \{\Phi_k\}_{k}^{\infty}\) are independence sets and of course partial order \(\leq\) up there in Definition 1 might be replaced by <.
The Definition 1 is the reason for calling Hasse digraph $D = \langle \Phi, \leq \rangle$ of the poset $(\Phi, \leq)$ a KoDAG as in Professor Kazimierz Kuratowski native language one word Komplet means complete ensemble – see more in [3] and for the history of this name see: The Internet Gian-Carlo Polish Seminar Subject 1. oDAGs and KoDAGs in Company (Dec. 2008).

Simultaneously – for the history of the Kwa´sniewski The Upside Down Notation Principle see: The Internet Gian-Carlo Polish Seminar Subject 2, upside down notation; leitmotiv: Is the upside down notation efficiency – an indication? of a structure to be named? (Feb. 2009).

**Definition 2.** Let $F = \langle k_F \rangle_{k=0}^n$ be an arbitrary natural numbers valued sequence, where $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. We say that the graded poset $P = (\Phi, \leq)$ is denominated (encoded=labelled) by $F$ iff $|\Phi_k| = k_F$ for $k = 0, 1, ..., n$. We shall also use the expression – "$F$-graded poset".

2. Combinatorial interpretation

For combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) called KoDAGs see [4, 5]. The recent equivalent formulation of this combinatorial interpretation is to be found in [4] (Feb 2009) or [6] from which we quote it here down.

**Definition 3.** $F$-nomial coefficients are defined as follows

\[
\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)F!} = \frac{n_F \cdot (n-1)_F \cdot \ldots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \ldots \cdot k_F}
\]

while

\[n, k \in \mathbb{N} \text{ and } 0_F! = n_0^F = 1 \text{ with } n_F^k = \frac{n_F!}{k_F!}
\]

staying for falling factorial. $F$ is called $F$-graded poset admissible sequence iff $(n_k)_F \in \mathbb{N} \cup \{0\}$ (In particular we shall use the expression – $F$-cobweb admissible sequence).

**Definition 4.**

\[C_{\max}(\Pi_n) \equiv \{c = \langle x_0, x_1, ..., x_n \rangle, x_s \in \Phi_s, s = 0, ..., n\}
\]

i.e. $C_{\max}(\Pi_n)$ is the set of all maximal chains of $\Pi_n$

and consequently (see Section 2 in [9] on Cobweb posets’ coding via $N^\infty$ lattice boxes).

**Definition 5.** $(C_{\max}^k, n)$ Let

\[C_{\max}(\Phi_k \rightarrow \Phi_n) \equiv \{c = \langle x_k, x_{k+1}, ..., x_n \rangle, x_s \in \Phi_s, s = k, ..., n\} \equiv \{\text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle\} \equiv C_{\max}(\langle \Phi_k \rightarrow \Phi_n \rangle) \equiv C_{\max}^{k, n}.
\]
Note. The

\[ C_{\max}(\Phi_k \rightarrow \Phi_n) \equiv C_{k,n}^{\max} \]

is the hyper-box points’ set [9] of Hasse sub-diagram corresponding maximal chains and it defines biunivoquely the layer \( \langle \Phi_k \rightarrow \Phi_n \rangle = \bigcup_{n=\Phi_s}^{\Phi_k} \Phi_s \) as the set of maximal chains’ nodes (and vice versa) – for these arbitrary \( F \)-denominated graded DAGs (KoDAGs included).

The equivalent to that of [4, 5] formulation of combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following.

Theorem 1 [6, 4]. (Kwa´sniewski) For \( F \)-cobweb admissible sequences \( F \)-nomial coefficient \( \binom{n}{k}_F \) is the cardinality of the family of equipotent to \( C_{\max}(P_m) \) mutually disjoint maximal chains sets, all together partitioning the set of maximal chains \( C_{\max}(\Phi_{k+1} \rightarrow \Phi_n) \) of the layer \( \langle \Phi_{k+1} \rightarrow \Phi_n \rangle \), where \( m = n - k \).

For environment needed and then simple combinatorial proof see [4, 5] easily accessible via Arxiv.

Comment 1. For the above Kwa´sniewski combinatorial interpretation of \( F \)-nomials’ array it does not matter of course whether the diagram is being directed or not, as this combinatorial interpretation is equally valid for partitions of the family of SimplePath\(_{\max}(\Phi_k - \Phi_n)\) in comparability graph of the Hasse digraph with self-explanatory notation used on the way. The other insight into this irrelevance for combinatoric interpretation is [9]: colligate the coding of \( C_{k,n}^{\max} \) by hyper-boxes. (More on that soon). And to this end recall what really also matters here: a poset is graded if and only if every connected component of its comparability graph is graded. We are concerned here with connected graded graphs and digraphs.

For the relevant recent developments see [7] while [8] is their all source paper as well as those reporting on the broader research (see [9–20, 22–26] and references therein). The inspiration for “philosophy” of notation in mathematics as that in Knuth’s from [21] – in the case of “upside-downs” has been driven by Gauss “\( q \)-natural numbers” \( \equiv N_q = \{ n_q = q^0 + q^1 + ... + q^{n-1} \}_{n \geq 0} \) from finite geometries of linear subspaces lattices over Galois fields. As for the earlier use and origins of the use of this author’s upside down notation see [27–43].

Comment 2. Colligate any binary relation \( R \) with Hasse digraph cover relation \( \preceq \cdot \) and identify as in SNAC \( \zeta(R) \equiv R^* \) with incidence algebra zeta function and with zeta matrix of the poset associated to its Hasse digraph, where

The reflexive reachability relation \( \zeta(R) \equiv R^* \) is defined as

\[ R^* = R^0 \cup R^1 \cup R^2 \cup ... \cup R^n \cup ... \bigcup_{k>0} R^k = R^\infty \cup I_A = \]

= transitive and reflexive closure of \( R \)

\[ \iff A(R^\infty) = A(R)^{\oplus 0} \lor A(R)^{\oplus 1} \lor A(R)^{\oplus 2} \lor ... \lor A(R)^{\oplus n} \lor ... , \]
Graded posets inverse zeta matrix formula I

where $A(R)$ is the Boolean adjacency matrix of the relation $R$ simple digraph and $\oplus$ stays for Boolean product.

Then colligate and/or recall from SNACK the resulting schemes.

**Schemes:**

\[
\leq = \leq^* = \text{connectivity of } \prec \cdot \\
\leq = \leq^* = \text{reflexive reachability of } \prec \cdot \\
\prec \cdot^* = \zeta(\prec \cdot).
\]

**Remark 1.** Obvious. Needed also for the next Section. Compare with the Observation 3 below.

The $\zeta$ matrix (≡ the algebra structure coding element of the incidence algebra $I(P, \leq)$) is the characteristic function $\chi$ of a partial order relation $\leq$ for any given $F$-graded poset including $F$-cobweb posets II:

\[
\zeta = \chi(\leq).
\]

The consequent (customary-like notation included) notation of other algebra $I(P, \leq)$ important elements then – for the any fixed order $\leq$ – is the following [1–3]:

\[
\zeta = \chi(\leq) = \zeta< + \delta, \\
\zeta< = \chi(<) = \zeta - \delta \equiv \rho, \ (\text{reachability = connectivity}), \\
\zeta< = \chi(\prec \cdot) \equiv \kappa, \ (\text{cover}), \\
\zeta\leq = \chi(\leq \prec \cdot) = \kappa + \delta \equiv \eta, \ (\text{reflexive"cover"})
\]

\[
\eta = \kappa + \delta = \begin{bmatrix}
I_1 & B_1 & \text{zeros} \\
I_2 & B_2 & \text{zeros} \\
I_3 & B_3 & \text{zeros} \\
\vdots & \vdots & \vdots \\
I_n & B_n & \text{zeros}
\end{bmatrix}
\]

\[
\eta^{-1} = (\delta + \kappa)^{-1} = \sum_{k \geq 0} (-\kappa)^k = \begin{bmatrix}
I_1 & -B_1 & B_1B_2 & \ldots \\
I_2 & -B_2 & \ldots \\
I_3 & -B_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots \\
I_n & -B_n & \ldots
\end{bmatrix}.
\]

Recall from SNACK: $B(A)$ is the biadjacency i.e cover relation $\prec \cdot$ matrix of the adjacency matrix $A$.

**Note.** Biadjacency and cover relation $\prec \cdot$ matrix for bipartite digraphs coincide. By extension – we call cover relation $\prec \cdot$ matrix $\kappa$ the biadjacency matrix too in order to keep reminiscent convocations going on.
As a consequence – quoting SNACK – we have:

\[ B(\oplus_{i=1}^{n} G_i) \equiv B[\oplus_{i=1}^{n} A(G_i)] = \oplus_{i=1}^{n} B[A(G_i)] \equiv \text{diag}(B_1, B_2, ..., B_n) = \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & \cdots & \cdots & B_n \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \cdots & \cdots & B_n \\ & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \]

or equivalently

\[ \kappa \equiv \chi(\ominus_{i=1}^{n} \prec \cdot \cdot \cdot) \equiv \begin{bmatrix} 0 & B_1 & & & & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & B_2 & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \]

\[ n \in N \cup \{ \infty \}. \]

In view of the all above the following is obvious;

\[ (A \oplus B)^{-1} \neq A^{-1} \oplus B^{-1} \]

except for the trivial case.

Anticipating considerations of Section III and customarily allowing for the identifications: \( \chi(\prec \cdot \cdot \cdot) \equiv \prec \cdot \cdot \cdot \equiv \kappa \) – consider \([\text{Max}] \in I(P, R)\):

\[ [\text{Max}] = (I - \prec \cdot \cdot \cdot)^{-1} = \sum_{k \geq 0} \kappa^k \]

in order to note that \((x \in \Phi_t \equiv x = x_t \in \Phi_t), [\text{Max}]_{s,t} = \text{the number of all maximal chains in the poset interval} \([x_s, x_t] \equiv [s, t] \text{ where} x_s \in \Phi_s \text{ and} x_t \in \Phi_t \text{ for}, \text{ say}, s \leq t \text{ with the reflexivity (loop) convention adopted i.e.} [\text{Max}]_{t,t} = 1.\]

**Sub-Remark 1.1.** It is now a good – prepared for – place to note further relevant properties of constructs as to be used in the sequel. These are the following.

\( C_{\text{max}}(\langle \Phi_r \to \Phi_k \rangle \oplus \langle \Phi_k \to \Phi_s \rangle) = C_{\text{max}}(\Phi_r \to \Phi_s), \)

for \( r \leq k \leq s \) while \(|\Phi_n| \equiv n_F.\)

Let \( |C_{\text{max}}| \equiv C^{k, n}. \) Then for \( F\)-cobweb posets (what about just \( F\)-graded?) we note that

\[ C^{r,k}C^{k,s} = k_F C^{r,s}, \]

hence

\[ C^{r,k}C^{k,s} = C^{r,s} \text{ if } k_F = 1 \]

for \( r \leq k \leq s \) while

\[ C^{r,k}C^{k+1,s} = C^{r,s} \]
for \( r \leq k < s \) while \(|\Phi_n| \equiv n_F\). Let us now see in more detail how this kind (Q.M.?) of mimics of Markov property is intrinsic for natural joins of digraphs. For that to do consider levels i.e. independent (stable) sets \( \Phi_k = \{ x_{k,i} \}_{i=1}^{k_F} \) and extend the notation accordingly so as to encompass
\[
\langle \Phi_r \to x_{k,i} \rangle = \{ c = x_r, x_{r+1}, \ldots, x_{k-1}, x_{k,i} \}, \quad x_s \in \Phi_s, \quad s = r, \ldots, k - 1 \}.
\]
Let
\[
|C_{\text{max}}(\Phi_r \to x_{k,i})| \equiv C^{r,k,i}_r.
\]
Then
\[
\sum_{i=1}^{k_F} C^{r,k,i}_r C^{s,k,i}_s = C^{r,s}_r
\]
for \( r \leq k < s \), ... (for \( r \leq k \leq s \) ?). In the case of cobweb posets (what about just \( F \)-graded?) the numbers \( C^{r,k,i}_s \) are the same for each \( i = 1, \ldots, k_F \) therefore we have for cobwebs
\[
k_F C^{r,k,i}_r C^{s,k,i}_s = C^{r,s}_r
\]
which in view of \( k_F C^{r,k,i}_r = C^{r,k}_r \) is of course consistent with \( C^{r,k} C^{k,s}_r = k_F C^{r,s}_r \). We consequently notice that – with self-evident extension of notation:
\[
\langle \Phi_k \to \Phi_n \rangle = \bigcup_{i,j=1}^{k_F,n_F} (x_{k,i} \to x_{n,j}).
\]

The frequently used block matrices are: 1) \( I(s \times k) \) which denotes \((s \times k)\) matrix of ones i.e. \( I(s \times k) \) is a \((s \times k)\) matrix with \( ij \)-th entry being 1; \( 1 \leq i \leq s, 1 \leq j \leq k \). and \( n \in N \cup \{ \infty \} \), 2) and \( B(s \times k) \) which stays for \((s \times k)\) matrix of ones and zeros accordingly to the \( F \)-graded poset has been fixed – see Observation 2.

In the block matrices language the above Markov property for cobweb posets (what about just \( F \)-graded?) reads as follows (to be used in Section 2) for example:
\[
I(r_F \times (r+1)_F) I ((r+1)_F \times (r+2)_F) = (r+1)_F I (r_F \times (r+2)_F).
\]
Well, what about then just \( F \)-graded? – see Comment 3 and its Warning.

Comment 3. Colligate and make identifications of graded DAGs with \( n \)-ary relations as in SNAC:
\[
\leq \Phi_0 \times \Phi_1 \times \ldots \times \Phi_n \iff \text{cobweb poset} \iff \text{KoDAG},
\]
for the natural join of di-bicliques and similarly for \( \leq \) being natural join of any sequence binary relations
\[
\leq \Phi_0 \times \Phi_1 \times \ldots \times \Phi_n \iff \text{cobweb poset} \iff \text{KoDAG}.
\]

Warning. Note that not for all \( F \)-graded posets their partial orders may be consequently identified with \( n \)-ary relations, where \( F = \langle k_F \rangle_{k=1}^n \) while \( n \in N \cup \{ \infty \} \). This is possible iff no biadjacency matrices entering the natural join for \( \leq \) has a zero column or a zero row. If a vertex \( m \in \Phi_k \) has not either incoming or outgoing arcs then we shall call it the mute node. This naming being adopted we may say now:
A. K. Kwaśniewski

A \textit{F}-graded poset may be identified with $n$-ary relation as above iff it is $F$-graded poset with no mute nodes. Equivalently – zero columns or rows in biadjacency matrices are forbidden. See and compare figures below.

![Fig. 1: Display of the natural join $\oplus \rightarrow$ of binary relations bipartite digraphs.](image1)

![Fig. 2: Display of the natural join $\oplus \rightarrow$ bipartite digraphs with a mute node.](image2)

![Fig. 3: Display of the layer $\langle \Phi_1 \rightarrow \Phi_4 \rangle = \text{the subposet } \Pi_4 \text{ of the } F = \text{Gaussian integers sequence } (q = 2) \text{ } F\text{-cobweb poset and } \sigma \Pi_4 \text{ subposet of the } \sigma \text{ permuted Gaussian } (q = 2) \text{ } F\text{-cobweb poset.}](image3)

3. \textbf{Examples of } $\zeta(\leq) \in I(\Pi, \mathbb{Z})$

Let $F$ denotes arbitrary natural numbers valued sequence. Let $A_N$ be the Hasse matrix i.e. adjacency matrix of cover relation $\prec \cdot$ digraph denominated by sequence $N$ [1]. Then the zeta matrix $\zeta = (1 - A_N)^{-1} \odot$ for the denominated by $F = N$ cobweb poset is of the form [1] (see also [15–20, 4]):
Graded posets inverse zeta matrix formula

Example 1. \( \zeta_N \). The incidence matrix for the N-cobweb poset. Note that the matrix \( \zeta \) representing uniquely its corresponding cobweb poset does exhibits a staircase structure of zeros above the diagonal (see above, see below) which is characteristic to Hasse diagrams of all cobweb posets and for graded posets it is characteristic too.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

Example 2. \( \zeta_F \). The matrix \( \zeta \) for the Fibonacci cobweb poset associated to F-KoDAG Hasse digraph.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]
The above remarks are visualized as below [15–20, 4]. Namely – apart from $F$-label, the another label and simultaneously visual code of cobweb graded poset is its “La scala” descending down there to infinity with picture which looks like that below.

Example 3. La Scala di Fibonacci. The staircase structure of incidence matrix $\zeta_F$ for $F=\text{Fibonacci sequence}$.
Note. The picture above is drawn for the sequence $F = \langle F_1, F_2, F_3, ..., F_n, ... \rangle$, where $F_k$ are Fibonacci numbers.

Description of the Figure “La Scala di Fibonacci” following [15–20, 4]. If one defines (see: [15–20] and for earlier references therein as well as in all [1–8]) the Fibonacci poset $\Pi = \langle P, \leq \rangle$ with help of its incidence matrix $\zeta$ representing $P$ uniquely then one arrives at $\zeta$ with easily recognizable staircase-like structure – of zeros in the upper part of this upper triangle matrix $\zeta$. This structure is depicted by the Figure “La Scala di Fibonacci” where: empty places mean zero values (under diagonal) and filled with - places mean values one (above the diagonal).

Advice. Simultaneous perpetual Exercises. How the all above and coming figures, formulas and expressions change (simplify) in the case of $2^{(1)}$ replacing the ring $\mathbb{Z}$ of integers in $I(\Pi, \mathbb{Z})$.

Comment 4. The given $F$-denominated staircase zeros structure above the diagonal of zeta matrix $\zeta$ is the unique characteristics of its corresponding $F$-KoDAG Hasse digraphs, where $F$ denotes any natural numbers valued sequence as shown below.

For that to deliver we use the Gaussian coefficients inherited upside down notation i.e. $F_n \equiv n_F$ (see [1–16], [27-30], and the Appendix in [9] extracted from [32]) and recall the Upside Down Notation Principle.

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Let us also easier the portraying task putting \( n_F = 1 \). Then – apart from \( F \)-label, the another label and simultaneously visual code of cobweb graded poset is its “La scala” descending down there to infinity with picture which looks like that below, where recall the \( F = (k_F)^n_{\geq 0} \) is an arbitrary natural numbers valued sequence finite or infinite as \( n \in N \cup \{0\} \cup \{\infty\} \).

Another special case Example is delivered by the Fig. 5 below.

Example 5. \( \zeta_F \). The matrix \( \zeta \) for \( (0_F = 1_F = 1 \) and \( n_F = 3 \) for \( n \geq 2 \) the special sequence \( F \) constituting the label sequence denominating cobweb poset associated to \( F \)-KoDAG Hasse digraph.

Advice. Simultaneous perpetual Exercises. How the all above and coming picture Examples, figures, formulas and expressions change (simplify) in the case of \( 2^{[1]} \) replacing the ring \( Z \) of integers in \( I(\Pi, Z) \).

4. Graded posets’ \( \zeta \) matrix formula

Recall now following SNACK that any graded poset with the finite set of minimal elements is an \( F \)-sequence denominated sub-poset of its corresponding cobweb poset.

The Observation 2 in SNACK supplies the simple recipe for the biadjacency (reduced adjacency) matrix of Hasse digraph coding any given graded poset with the finite set of minimal elements. The recipe for zeta matrix is then standard. We illustrate this by the SNACK source example; the source example as the adjacency matrices i.e. zeta matrices of any given graded poset with the finite set of minimal
elements are sub-matrices of their corresponding cobweb posets and as such have the same block matrix structure and differ “only” by eventual additional zeros in upper triangle matrix part while staying to be of the same cobweb poset block type.

The explicit expression for zeta matrix \( \zeta_F \) of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued \( F \)-sequence was given in \cite{1} due to more than mnemonic efficiency of the up-side-down notation being applied (see \cite{1} and references therein). With this notation inspired by Gauss and replacing \( k \)-natural numbers with “\( k_F \)” numbers (Note. The Upside Down Notation Principle has been used in \cite{1}) one gets:

\[
A_F = \begin{bmatrix}
0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_{n_F \times 1_F} & 0_{n_F \times 2_F} & 0_{n_F \times 3_F} & 0_{n_F \times 4_F} & 0_{n_F \times 5_F} & \cdots \\
\end{bmatrix}
\]

and

\[
\zeta_F = \exp[1 - A_F] = (1 - A_F)^{-1} \equiv I_{\infty \times \infty} + A_F + A_F^2 + \cdots = \\
\begin{bmatrix}
I_{1_F \times 1_F} & I(1_F \times \infty) \\
O_{2_F \times 1_F} & O_{2_F \times 2_F} & I(2_F \times 3_F) \\
O_{3_F \times 1_F} & O_{3_F \times 2_F} & O_{3_F \times 3_F} & I(3_F \times 4_F) \\
O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & O_{4_F \times 4_F} & I(4_F \times 5_F) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
O_{n_F \times 1_F} & O_{n_F \times 2_F} & O_{n_F \times 3_F} & O_{n_F \times 4_F} & O_{n_F \times 5_F} & \cdots \\
\end{bmatrix}
\]

where \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \( I(s \times k)_{ij} = 1 \), \( 1 \leq i \leq s \), \( 1 \leq j \leq k \) and \( n \in N \cup \{ \infty \} \).

Particular examples of the above block structure of \( \zeta \) matrix (resulting from \( \zeta \) being a result of natural join operations on the way) are supplied by Examples 1, 2, 3, 4, 5 above and Examples 6, 7, 8 represented by Fig. 4, Fig. 5, Fig. 6 below. As a matter of fact – all elements \( \sigma \) of the incidence algebra \( I(P, R) \) including \( \zeta \) i.e. characteristic function of the partial order \( \leq \) or Möbius function \( \mu = \zeta^{-1} \) (as exemplified with Examples 9, 10, 11, 12 below) have the same block structure encoded by \( F \) sequence chosen. Recall that \( R \) from \( I(P, R) \) denotes commutative ring and for example \( R \) might be taken to be Boolean algebra \( 2^{(1)} \), the field \( Z_2 = \{0, 1\} \) the ring \( Z \) of integers or real or complex or \( p \)-adic numbers.

Namely, arbitrary \( \sigma \in I(P, R) \) is of the form

\[
\sigma = \begin{bmatrix}
D_{1_F \times 1_F} & M(1_F \times \infty) \\
O_{2_F \times 1_F} & D_{2_F \times 2_F} & M(2_F \times \infty) \\
O_{3_F \times 1_F} & O_{3_F \times 2_F} & D_{3_F \times 3_F} & M(3_F \times \infty) \\
O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & D_{4_F \times 4_F} & M(4_F \times \infty) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_{k_F \times k_F} & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

where \( D_{k_F \times k_F} \) denotes diagonal \( k_F \times k_F \) matrix while \( M(n_F \times \infty) \) stays for arbitrary \( n_F \times \infty \) matrix and both with matrix elements from the ring \( R = 2^{(1)} \), \( Z_2 = \{0, 1\} \), \( Z \) etc.
In more detail: it is trivial to note that all elements \( \sigma \in I(P,R) \) – including \( \zeta^{-1} \) for which \( D_{k_F \times k_F} = I_{k_F \times k_F} \) – are of matrix block form resulting from \( \oplus \rightarrow \) of the subsequent bipartite digraphs \( (\Phi_k \cup \Phi_{k+1},R) \), \( R \subseteq \Phi_k \times \Phi_{k+1} \), \( |\Phi_k| = k_F \) i.e.

\[
\sigma = \begin{bmatrix}
D_{1_F \times 1_F} & M(1_F \times 2_F) & M(1_F \times 3_F) & M(1_F \times 4_F) & M(1_F \times 5_F) & M(1_F \times 6_F) \\
0_{2_F \times 1_F} & D_{2_F \times 2_F} & M(2_F \times 3_F) & M(2_F \times 4_F) & M(2_F \times 5_F) & M(2_F \times 6_F) \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & D_{3_F \times 3_F} & M(3_F \times 4_F) & M(3_F \times 5_F) & M(3_F \times 6_F) \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & D_{4_F \times 4_F} & M(4_F \times 5_F) & M(4_F \times 6_F) \\
... & etc & ... & & & \end{bmatrix}
\]

where \( M(k_F \times (k + 1)_F) \) denote corresponding \( k_F \times (k + 1)_F \) matrices with matrix elements from the ring \( R = 2^{(1)} \), \( Z_2 = \{0, 1\} \), \( Z \) etc. However... for some seemingly most useful of them...

The New Name: \( \oplus \rightarrow \) natural \( \equiv M(k_F \times (k + 1)_F), r_s = c_{r,s}B(k_F \times (k + 1)_F)_{r,s} \),

In the case of \( \zeta \) or August Ferdinand Möbius matrices motivating examples of specifically natural elements \( \sigma \in I(P,R) \) (i.e. \( \oplus \rightarrow \) natural including those obtained via the ruling formula) – so in the case of such type elements \( \sigma \in I(P,R) \) we ascertain – and may prove via just see it – that

\[
M(k_F \times (k + 1)_F), r_s = c_{r,s}B(k_F \times (k + 1)_F)_{r,s},
\]

where the rectangular “zero-one” \( B(k_F \times (k + 1)_F) \) matrices from Observation 2. are obtained from the \( F \)-cobweb poset matrices \( I(k_F \times (k + 1)_F) \) by replacing some ones by zeros.

Moreover (see Observation 3) – in the case of Möbius \( \mu = \zeta^{-1} \) matrix as it is obligatory \( c_{r,r+1} = -1 \).

The motivating example of \( \oplus \rightarrow \) natural element of the incidence algebra is \( \zeta_F \) due to the algorithm of the ruling formula considered over the \( R = 2^{(1)} \) ring in this particular case element:

\[
\zeta_F = I_{\infty \times \infty} + A_F + A_F^{\oplus 2} + ... = (1 - A_F)^{-1\oplus}
\]

where

\[
A_F = \begin{bmatrix}
0_{1_F \times 1_F} & B(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & B(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & B(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & B(4_F \times 5_F) \end{bmatrix}
\]

and where \( B(k_F \times (k + 1)_F) \) are introduced by the Observation 2.

For the other example of \( \oplus \rightarrow \) natural element is \([Max]\) given by the algorithm of the ruling formula over the \( R = Z \) ring see further on in below.

For the sake of the forthcoming Observation 1 we introduce the set of corresponding Hasse diagram maximal chains called the layer of the graded DAG called KoDAG to be just this \([8, 6, 5, 4]\):
\[\langle \Phi_k \rightarrow \Phi_n \rangle = \{c = \langle x_k, x_{k+1}, \ldots, x_n \rangle, x_s \in \Phi_s, s = k, \ldots, n\} \]

**Observation 1.** [SNACK – and consult the Remark 1 above.] Let us denote by \(\langle \Phi_k \rightarrow \Phi_{k+1} \rangle\) the di-bicliques denominated by subsequent levels \(\Phi_k, \Phi_{k+1}\) of the graded \(F\)-poset \(P(D) = (\Phi, \leq)\) i.e. levels \(\Phi_k, \Phi_{k+1}\) of its cover relation graded digraph \(D = (\Phi, \prec)\) [Hasse diagram]. Then

\[
B(\oplus_{k=1}^n \langle \Phi_k \rightarrow \Phi_{k+1} \rangle) = \text{diag}(I_1, I_2, \ldots, I_n)
\]

where \(I_k \equiv I(kF \times (k+1)F), k = 1, \ldots, n\) and where – recall – \(I(s \times k)\) stays for \((s \times k)\) matrix of ones i.e. 
\[
I_{ij} = \begin{cases} 
1 & \text{if } 1 \leq i \leq s, 1 \leq j \leq k. \\
0 & \text{otherwise}. 
\end{cases}
\]

The binary natural join operation \(\oplus\rightarrow\) being defined for such pairs of arguments (matrices, digraphs, graphs, relations of varying arity,...) which do satisfy the natural join condition (see SNACK and [3, 2]) is associative of course iff performable and obviously \(\oplus\rightarrow\) is noncommutative.

The recipe for any connected – hence \(F\)-denominated – the recipe for any given graded poset with a finite minimal elements set is supplied via the following observation.

**Observation 2.** [SNACK – and consult the Remark 1 above]. Consider bigraphs’ chain obtained from the above di-bicliques’ chain via deleting or no arcs making thus [if deleting arcs] some or all of the di-bicliques \(\langle \Phi_k \rightarrow \Phi_{k+1} \rangle\) not di-bicliques; denote them as \(G_k\). Let \(B_k = B(G_k)\) denotes their biadjacency matrices correspondingly. Then for any such \(F\)-denominated chain [hence any chain] of bipartite digraphs \(G_k\) the general formula is:

\[
B(\oplus_{i=1}^n G_i) \equiv B[\oplus_{i=1}^n A(G_i)] = \oplus_{i=1}^n B[A(G_i)] \equiv \text{diag}(B_1, B_2, \ldots, B_n) =
\]

\[
\begin{bmatrix}
B_1 & & \\
& B_2 & \\
& & \ddots \\
& & & B_n
\end{bmatrix}
\]

\(n \in \mathbb{N} \cup \{\infty\}\).

Let us recall that \(\zeta\) is defined for any poset as follows \((p, q \in P)\):

\[
\zeta(p, q) = \begin{cases} 
1 & \text{for } p \leq q, \\
0 & \text{otherwise}. 
\end{cases}
\]
This is the reason why in the above ruling formula:

\[ \zeta_F = I_{\infty \times \infty} + A_F + A_F^{\otimes 2} + \ldots = (1 - A_F)^{-1} \]

the Boolean powers are used. If this rule is applied with Z-ring or other ring \( R, Z \subseteq R \) powers then we get

\[
[Max]_F = A_F^0 + A_F^1 + A_F^2 + \ldots = (1 - A_F)^{-1} = \\
\begin{bmatrix}
I_{1_F \times 1_F} & B(1_F \times 2_F) & B(1_F \times 3_F) & B(1_F \times 4_F) & B(1_F \times 5_F) & \ldots \\
0_{1_F \times 1_F} & I_{2_F \times 2_F} & B(2_F \times 3_F) & B(2_F \times 4_F) & B(2_F \times 5_F) & \ldots \\
0_{1_F \times 1_F} & 0_{1_F \times 2_F} & I_{3_F \times 3_F} & B(3_F \times 4_F) & B(3_F \times 5_F) & \ldots \\
0_{1_F \times 1_F} & 0_{1_F \times 2_F} & 0_{1_F \times 3_F} & I_{4_F \times 4_F} & B(4_F \times 5_F) & \ldots \\
\ldots & \ldots & \ldots & \ldots & \text{etc} & \ldots \\
\end{bmatrix}
\]

where \( B(k_F \times (k + 1)_F) \) are introduced by the Observation 2.

It is a matter of simple observation and induction to see that

\[ B(r_F \times (r + 2)_F) = B(r_F \times (r + 1)_F)B((r + 1)_F \times (r + 2)_F) \]

and consequently for \( s > r + 2 \)

\[ B(r_F \times s_F) = B(r_F \times (r + 1)_F)B((r + 1)_F \times (r + 2)_F) \ldots \]

\[ \ldots B((s - 2)_F \times (s - 1)_F)B((s - 1)_F \times s_F). \]

In the case of \( F \)-cobweb posets – replace \( B(r_F \times s_F) \) by \( I(r_F \times s_F) \) and then one may use the “Markov” property.

What about then just \( F \)-graded posets case? – See Comment 3 and its Warning.

**Remark 2.** \( F \)-graded poset construction – summary. The knowledge of \( \zeta \) matrix explicit form enables one to construct (calculate) via standard algorithms the Möbius matrix \( \mu = \zeta^{-1} \) and other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of \( P \). Right from the definition of \( P \) via its Hasse diagram. The way the \( \zeta \) is written above underlines the fact that this is the staircase structure encoding formula for any natural numbers valued sequence \( F \). Recall: this sequence \( F \) serves as the label encoding all resulting digraphs and combinatorial objects.

The subsequent di-biclique of bipartite digraph adjoining via natural join \( \oplus \rightarrow \) is in one to one correspondence with adjoining another subsequent one step down of La Scala. In another words: one more step down La Scala – one more di-biclique \( \oplus \rightarrow \)– adjoint.

To this end define the \( L \)-Logic function as follows:

\[ L([Max]_F) = \zeta_F, \quad \zeta_{r,s} = \begin{cases} 1 & [Max]_{r,s} > 0 \\ 0 & [Max]_{r,s} = 0 \end{cases} \]

This completes the natural join \( \oplus \rightarrow \) structural description of \( \zeta_F \) matrix construction for any \( F \)-graded poset and will be of use as a guide while looking for the similar form of Möbius matrix \( \mu = \zeta^{-1} \) bearing in mind that for \( s > r \)
\[ B(r_F \times s_F) = \prod_{i=r}^{s-1} B(i_F \times (i+1)_F). \]

**Remark 3.** The choice of \( F \)-poset II labeling and then Knuth notation. If one defines any graded \( F \) poset \( P \) with help of its incidence matrix \( \zeta \) representing \( P \) uniquely then in case of cobweb posets one arrives at \( \zeta \) with Type characterization La Scala code of zeros in the upper part of this upper triangle matrix \( \zeta \) due to implicit natural for right-handed oriented choice of nodes labeling. See all figures above. In the case of arbitrary \( F \)-graded poset \( P \) apart from La Scala additional zeros appear. These are the fixed zeros of \( B(i_F \times (i+1)_F) \) yielding all the other zeros from \( B(r_F \times s_F) \) in the upper block triangle of \( \zeta \) matrix via the product formula above. Let us make now this choice of labeling *explicit*. For that to do it is enough to focus on any cobweb poset II as a sample case.

**Remark 3.1.** A bit of history. The matrix elements of \( \zeta(x, y) \) matrix for Fibonacci cobweb poset were given in 2003 ([16, 20] Kwaśniewski) using \( x, y \in N \cup \{0\} \) labels of vertices in their “natural” linear order:

1. set \( k = 0 \),
2. then label subsequent vertices – from the left to the right – along the level \( k \),
3. repeat 2 for \( k \to k + 1 \) until \( k = n + 1 \); \( n \in N \cup \{\infty\} \).

As the result we obtain the \( \zeta \) matrix for Fibonacci sequence as presented by the the Fig. *La Scala di Fibonacci* dating back to 2003 [16, 20].

**Comment 0.** Mantra needed. If the statement \( s(F) \) depends (relies, is based on, “lies in ambush”) only on the fact that \( F \) is a natural valued numbers sequence then if the statement \( s(F) \) is proved true for \( F = N \) then it is true for any natural valued numbers sequence \( F \).

The origin – of effectiveness. Inspired [27–29, 32–38] by Gauss \( n_q = q^0 + q^1 + \ldots + q^{n-1} \) finite geometries numbers and in the spirit of Knuth “notationlogy” [21] we shall refer here also to the upside down notation effectiveness as in [1–4] or earlier in [27–30, 32–38], (specifically consult [32]). As for that upside down attitude \( F_n \equiv n_F \) being much more than “just a convention” to be used substantially in what follows as well as for the reader’s convenience – let us recall it just here quoting it as The Principle according to Kwaśniewski [4] (Feb 2009) where this rule has been formulated as an “of course” Principle i.e. simultaneously trivial and powerful statement.

**The Upside Down Notation Principle**

1. Let the statement \( s(F) \) depends only on the fact that \( F \) is a natural numbers valued statement.

2. Then if one proves that \( s(N) \equiv s((n)_{n \in N}) \) is true – the statement \( s(F) \equiv s((n_F)_{n \in N}) \) is also true. Formally – use equivalence relation classes induced by co-images of \( s : \{F\} \mapsto 2^{(1)} \) and proceed in a standard way.
In order to proceed further let us now recall-rewrite purposely here Kwaśniewski 2003 – formula for \( \zeta \) function of arbitrary cobweb poset so as to see that its' algorithm rules automatically make it valid for all \( F \)-cobweb posets where \( F \) is any natural numbers valued sequence i.e. with \( F_0 > 0 \). \( I(\Pi, R) \) stays for the incidence algebra of the poset \( \Pi \) over the commutative ring \( R \) where \( x, y, k, s \in \mathbb{N} \cup \{0\} \)

\[
\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y)
\]

\[
\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y)
\]

\[
\zeta_0(x, y) = \sum_{k \geq 1} \sum_{s \geq 0} \delta(x, F_{s+1} + k) \sum_{r=1}^{F_s - k - 1} \delta(k + F_{s+1} + r, y)
\]

and naturally

\[
\delta(x, y) = \begin{cases} 
1 & x = y \\
0 & x \neq y
\end{cases}
\]

The above formula for \( \zeta \in I(\Pi, R) \) rewritten in \( (F_s \equiv s_F) \) upside down notation equivalent form as below is of course valid for all cobweb posets \( x, y, k, s \in \mathbb{N} \cup \{0\} \).

\[
\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y), \quad \zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y),
\]

\[
\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r=1}^{(s-1)F - k - 1} \delta(x + r, y).
\]

Note. \( +\zeta_1 \) “produces the Pacific ocean of 1’s” in the whole upper triangle part of a would be incidence algebra \( \sigma \in I(\Pi, R) \) matrix elements with then \( -\zeta_0 \) resulting zeros and ones multiplying arbitrary \( \sigma \) choice fixed elements of \( R \).

Note. \( -\zeta_0 \) cuts out 0’s i.e. thus producing “zeros’ \( F \)-La Scala staircase” in the 1’s delivered by \( +\zeta_1 \).

This results exactly in forming 0’s rectangular triangles: \( s_F - 1 \) of them at the start of subsequent stair and then down to one 0 till – after \( s_F - 1 \) rows passed by one reaches a half-line of 1’s which is running to the right-right to infinity and thus marks the next in order stair of the \( F \)-La Scala.

The \( \zeta \) matrix explicit formula was given for arbitrary graded posets with the finite set of minimal in terms of natural join of bipartite digraphs in SNACK = the Sylvester Night Article on KoDAGs and Cobwebs = [1].

Recall 1. Recapitulation – the La Scala Mantra. What was said is equivalent to the fact that the cobweb poset coding La Scala is of the natural join operation origin thus producing \( \zeta \) matrix [1–3] with one down step of La Scala being equivalent to \( \oplus \rightarrow \)-adjoining the subsequent bipartite digraph and what results in: (quote from SNACK = [1], see: Subsection 2.6).
The explicit expression for zeta matrix $\zeta_F$ of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued $F$-sequence was given in [1] due to more than mnemonic efficiency of the up-side-down notation being applied (see [4] [v6] Feb 2009 and references therein). With this notation inspired by Gauss and replacing $k$-natural numbers with “$k_F$” numbers one gets

$$A_F = \begin{bmatrix}
0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}$$

and

$$\zeta_F = \exp_\oplus[A_F] \equiv (1 - A_F)^{-1} \equiv I_{\infty \times \infty} + A_F + A_F^\oplus + \ldots = \begin{bmatrix}
I_{1_F \times 1_F} & I(1_F \times \infty) \\
O_{2_F \times 1_F} & O_{2_F \times 2_F} & I(2_F \times \infty) \\
O_{3_F \times 1_F} & O_{3_F \times 2_F} & O_{3_F \times 3_F} & I(3_F \times \infty) \\
O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & O_{4_F \times 4_F} & I(4_F \times \infty) \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}$$

where $I(s \times k)$ stays for $(s \times k)$ matrix of ones i.e. $[I(s \times k)]_{ij} = 1$; $1 \leq i \leq s$, $1 \leq j \leq k$. and $n \in N \cup \{\infty\}$.

In the $\zeta_F$ formula from [1–3] $\oplus$ denotes the Boolean product, hence – used for Boolean powers too. We readily recognize from its block structure that $F$-La Scala is formed by upper zeros of block-diagonal matrices $I_{k_F \times k_F}$ which sacrifice these their zeros to constitute the $k$-th subsequent stair in the $F$-La Scala descending and descending far away down to infinity. Thus the cobweb poset coding La Scala is due to the natural join origin of $\zeta$ matrix. In the general case of any $F$-graded poset with as in Remark 3.1 labeling fixed one naturally encounters apart from obligatory La Scala zeros generated via the ruling formula from (Remark.1.) those of $B(A)$ which is biadjacency i.e. cover relation $\prec \cdot$ matrix of the adjacency matrix $A$ of the $F$-graded poset.

**Note.** Biadjacency and cover relation $\prec \cdot$ matrix for bipartite digraphs coincide. By extension – we call cover relation $\prec \cdot$ matrix $\kappa$ the biadjacency matrix too in order to keep reminiscent convocations going on.

Note now that because of $\delta$’s under summations in the former $\zeta$ formula the following is obvious:

$$1 \leq r = y - x \leq (s - 1)_F - k - 1 \equiv 1 \leq r = y - k - s_F \leq s - 1)_F - k - 1 \equiv 1 \leq r = y \leq s_F - (s - 1)_F - 1.$$
\[ \zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y), \]

where
\[ \zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y), \]

[- note: \( +\zeta_1 \) “produces the Pacific ocean of 1’s” in the whole upper triangle part of a would be incidence algebra \( \sigma \in I(\Pi, R) \) matrix elements with then \( -\zeta_0 \) resulting zeros and ones multiplying arbitrary \( \sigma \) choice fixed elements of \( R \), and where (where \( x, y, k, s \in \mathbb{N} \cup \{0\} \))

\[ \zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x + s F, k + s F) \sum_{r \geq 1} \delta(r, y), \]

[- note then again that \( -\zeta_0 \) cuts out “one’s F-La Scala staircase” in the 1’s provided by \( +\zeta_1 \).

Note, that for \( F = \text{Fibonacci} \) this still more simplifies as then
\[ s F + (s - 1) F - 1 = (s + 1) F. \]

**Remark 3.2.** ad Knuth notation [21] indicated back to me by Maciej Dziemiańczuk. In the wise “notationology” note [21] one finds among others the notation just for the purpose here (see [4] v6 Fri, 20 Feb 2009)

\[ [s] = \begin{cases} 
1 & \text{if } s \text{ is true,} \\
0 & \text{otherwise.} 
\end{cases} \]

Consequently for any set or class
\[ [x = y] \equiv \delta(x, y). \]

Consequently for any set with addition (group, free group, semi-group, ring,...):
\[ [x < y] \equiv \sum_{k \geq 1} \delta(x + k, y), \]
\[ [x \leq y] \equiv \sum_{k \geq 0} \delta(x + k, y). \]

Using this makes my last above expression of the \( \zeta \) in terms of \( \delta \) still more transparent and handy if rewritten in Donald Ervin Knuth’s notation [21]. Namely:
\[ \zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y) \]
\[ \zeta_1(x, y) = [x \leq y] \]
\[ \zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} [x = k + s F][1 \leq y \leq s F + (s - 1) F - 1], \]
\[ \zeta_0(x, y) = \sum_{s \geq 1} [x > s F][1 \leq y \leq s F + (s - 1) F - 1], \]

where, let us recall: \( x, y, k, s \in \mathbb{N} \cup \{0\} \).
Note, that for \( F = \text{Fibonacci} \) this still more simplifies as then
\[
s_F + (s-1)F - 1 = (s+1)F.
\]

Remark 3.3. Knuth notation [21] – and Dziemiańczuk’s guess? It was remarked by my Gdańsk University Student Coworker Maciej Dziemiańczuk – that my \( \zeta \in I(\Pi, R) \) (equivalent) expressions are valid according to him only for \( F = \text{Fibonacci sequence} \).

In view of the Upside Down Notation Principle if any of these is proved valid for any particular natural numbers valued sequence \( F \) using no other particular properties of \( F \) then it should be true for all of the kind.

His this being doubtful – has led him to invention of his own – in the course of our The Internet Gian Carlo Rota Polish Seminar discussions with me (see [4] 20 Feb 2009).

Here comes the formula postulated by him in the course the The Internet Seminar e-mail discussions (see then resulting now Comment 5 referring to Krot).

\[
\zeta(x, y) = [x \leq y] - [x < y] \sum_{n \geq 0} ([x > S(n)]) [y \leq S(n+1)],
\]

where
\[
S(n) = \sum_{k \geq 1} kF.
\]

Exercise. My todays reply to his guess (compare [17] 20 Feb 2009) is the following Exercise.

Let \( x, y \in N \cup \{0\} \) be the labels of vertices in their “natural” linear order as explained earlier.

Prove the true claim: Dziemiańczuk guess is equivalent to Kwaśniewski formulas.

– What is for? My “for” is the Socratic Method question. Why not use the arguments in favor of

\[
\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + sF) \sum_{r \geq 1} \delta(r, y).
\]

Hint. Use the same argumentation. Hint. Then – contact Comment 5.

Here are some illustrative examples-exercises with pictures [Figures 4, 5, 6] delivered by Maciej Dziemiańczuk’s computer personal service using the above Dziemiańczuk guess (see Comment 5).

Remark 4. Ewa Krot Choice. While the above is established it is a matter of simple observation by inspection to find out how does the the Mőbius matrix \( \mu = \zeta^{-1} \) looks like. Using in [22, 23] this author example and expression for \( \zeta \) matrix this has been accomplished first (see also [23]) for Fibonacci sequence and then the same formula was declared to be valid for \( F \) sequences as above in [24, 25]. Namely the author of [26] states that the Mőbius function for the Fibonacci sequence designated cobweb
A. K. Kwaśniewski

Fig. 4: Example 6. Display of the $\zeta = 90 \times 90$. The subposet $\Pi_t$ of the $N$ i.e. integers sequence $N$-cobweb poset. $t =$?

The poset can be easily extended to the whole family of all cobweb posets with indication to the reference [24] where one neither finds the proof except for declaration that the validity for all cobweb posets is OK. From the today’s perspective the present author should say that it is not so automatic. For that to see follow what follows.

By now here is her formula for the cobweb posets’ Möbius function (see: (6) in [22] then it is recommended to consult Comment 5).

Let $x = \langle s, t \rangle$ and $y = \langle u, v \rangle$ where $1 \leq s \leq F_t$, $1 \leq u \leq F_v$ while $t, v \in N$.

These are descriptive and extra external with respect to the Krot formula below conditions imposed in order to stay in accordance with the zeros’ “La Scala di Fibonacci” structure of the present author “discovered” in 2003 [16, 17]. This Ewa Krot brave independence declaration step formula was since now on presented by the author of [23–26] in opposition (?) to the Kwaśniewski’s choice which makes these conditions being automatically inherited from $\zeta$ matrix formula by the present author (see 2003 [17] and consequently all relevant papers of Kwaśniewski later on till today).

If these external with respect to formula conditions are assumed then $\mu$ Möbius function for Fibonacci cobweb Krot formula reads [22]
Fig. 5: Example 7. Display of the $\zeta = 90 \times 90$. The subposet $\Pi_t$ of the $F = \text{Gaussian integers sequence (}q = 2\text{)}$. $F$-cobweb poset. $t = ?$

$$
\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle)
= \delta(s, u)\delta(t, v) - \delta(t + 1, v) + \sum_{k=2}^{\infty} \delta(t + k, v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1).
$$

In particular for Fibonacci sequence either $F = \langle 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \rangle$ or $F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \rangle$ we get the right number

$$
\mu(\langle 1, 1 \rangle, \langle 2, 1 \rangle) = -1, \quad \text{as} \quad \prod_{i=1+1}^{2} (F_i - 1) = \prod_{i=0}^{2} (F_i - 1) = 0.
$$

The same is right for $F = N$. We shall see also by inspection via Examples below that this is a obviously decisive sensitive starting point in applying the recurrent definition of Möbius function matrix $\mu$ and its descendant – the block structure of Möbius function coding matrix $C(\mu)$ – with this latter recurrence for $C(\mu)$ allowing simple solution simultaneously with combinatorial interpretation of Kroton matrix $K = (K_s(r_F))$, where $K_s(r_F) = |C(\mu)_{r,s}|$.

Now bearing in mind the Upside Down Notation Principle let start to prepare the formula for all connected graded posets ($F$-cobweb posets included) with $F_0 > 0$ (as it should be for natural numbers valued sequences) and of course for other natural
numbers valued sequences $F$. Note the condition resulting from $F_0 > 0$ unavoidable convention: Fibonacci means since now on $F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \rangle$.

At first the First Step. Let us formulate equivalent versions of the above Krot formula in coordinate grid $\mathbb{Z} \times \mathbb{Z}$ adequately to to the task of verifying it in the case of Fibonacci sequence $F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \rangle$. This has been done we arrive at what follows.

Let $x = \langle s, t \rangle$ and $y = \langle u, v \rangle$ where $1 \leq s \leq t_F$, $1 \leq u \leq v_F$ while $t, v \in \mathbb{N}$. Then (versions equivalent to Krot formula)

$$
\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = [(s = u)[t = v] - [t + 1 = v] + \sum_{k=2}^{\infty} [t + k = v](-1)^k \prod_{i=t+1}^{v-1} (i_F - 1)
$$

$$
\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = [(s = u)[t = v] - [t + 1 = v] + [t + 1 < v](-1)^{v-1} \prod_{i=t+1}^{u-1} (i_F - 1)
$$
or with sine qua non conditions being implemented in there:

\[
\mu(x, y) = \mu(s, v) = [(s = u) [t = v] - [t + 1 = v] + \\
[t + 1 < v] \sum_{1 \leq s \leq t_{F}} \sum_{1 \leq u \leq v_{F}} (-1)^{k} \prod_{i=t_{F}+1}^{v-1} (iF - 1).
\]

The above Möbius function re-formulas if proved valid for \( F = N \) thanks to no more than the assumption \( n_{N} \in N \) then it should be literally valid for all natural numbers valued sequences \( F \).

These formulas for Möbius function appear suitable [check] for Fibonacci sequence \( F = \{1, 2, 3, 5, 8, 13, 21, 34, \ldots\} \) as well [check] as in the case of \( F = N \) (Example 9.) and as well as in the case of Example 12 (see both below).

As a matter of fact this might be already expected from the following simple check using any of the equivalent formulas:

\[
\mu((1, 1), (2, 1)) = -1
\]

which is the right number for Fibonacci sequence (or see Example 11) as well as

\[
\mu((1, 1), (2, 1)) = (\mu((1, 1), (2, 2))) - 1
\]

is right the number for \( F = N \) natural number sequence (or see Example 12). The reason for that fact is at hand just by inspection of Hasse digraphs of cobweb posets under consideration. And these checks are crucial at the start in view of recurrent form of August Ferdinand Möbius matrix \( \mu \) formula. However...

However we are in need of The Proof! of this Krot Formula for Möbius function for any one – hence for all of the relevant sequences \( F \). Here let us call back (hail) the mantra: The Upside Down Notation Principle.

So – as now we see it – one is in need of the Second Step.

In the sequel this is to be done and we shall use formulas for Möbius function with the structure inferred from the fact that incidence algebra \( I(\Pi, R) \) elements arise in the sequential natural join of di-bicliques or bipartite digraphs in the general case of \( F \)-graded posets as to be exemplified and derived below. Then implementation of the recurrent definition of Möbius function matrix \( \mu \) gives birth to daughter descendant of \( \mu \) i.e. the block structure of Möbius function coding matrix \( C(\mu) \) implying for \( C(\mu) \) an recurrence allowing simple solution simultaneously with combinatorial interpretation of Kroton matrix \( K = (K_{s}(r_{F})) \), where \( K_{s}(r_{F}) = |C(\mu)_{r,s}| \). And this is to be this Second Step.

Before doing this in the next section – to this end – lets for now continue “the Krot and Krot-Sieniawiska contribution subject”. The author of [22] introduces parallely also another form of \( \zeta \) function formula and since now on – except for [24, 40] – in subsequent papers [23, 25, 26] their author uses the formula for \( \zeta \) function in this another form. Namely – this other form formula for \( \zeta \) function in the present authors’ grid coordinate system description of the cobweb posets was given by Krot in her note on Möbius function and Möbius inversion formula for Fibonacci cobweb poset [23] with \( F \) designating the Fibonacci cobweb posets. In [24] the formula the
Krot formula for the Möbius function for Fibonacci sequence $F$ was declared as valid for all cobweb posets i.e. for all natural numbers valued sequences $F$ denominated cobweb posets. (Consult also so the recent note “On Characteristic Polynomials of the Family of Cobweb Posets” [26] and see also Comment 5.).

Comment 5. ad $zeta$ and $\mu$. Back to 03 Feb 2004 preprint [40] for to see the source of the past future (?).

Aside explanatory gloss ad the above question mark (?)... the past future means future with respect to Feb 2004 till yesterday which is the interval of the past today – as expressed in Tachion language, where Tachion is also this author’s nickname apart from the meaning given to tachions by Feynman diagrams. The Past and The Future are relative in relativistic quantum theories still under construction since decades.

See for example [45] (“...The massive tensor particle is a tachion or a ghost depending... and we should compute those one-loop Feynman...”).

See for example [46] (“...See Simulating physics with computers by Feynman, IJTP 21 1982 [contact... John Archibald Wheeler to Tachion and then...]”).

Comment 5. ad $zeta$ and $\mu$. The meritum statement. In [40] – supervised by the present author (see Acknowledgments) the deliberate task was to consider just the case of Fibonacci sequence in order to to find the inverse matrix $zeta^{-1}$ of the $zeta$ from [15] (November 2003) using the present author $\zeta$ matrix expression in terms of the infinite Kronecker delta matrix $\delta$ from [16] (November 2003) and [17] (December 2003). Why Fibonacci? See the formula (5) page 9 in [40]. Applying (5) to the conditions on the top of the page 9 above the relevant formula (5) one arrives at (6) which afterwords – in coordinate grid description reads:

$$x = \langle s, t \rangle, \ y = \langle u, v \rangle, \ where \ 1 \leq s \leq F_t, \ 1 \leq u \leq F_v, \ while \ t, v \in \mathbb{N}.$$ 

Nevertheless already in Ewa Krot preprint [40] (see the top of the page 9 above the relevant formulas (5) and (6)) already there the general case conditions are stated which in notation of the present author labeling and upside down notation as well as due to Dziemiańczuk’s observed Knuth notation now simply read as follows:

$$[(x > S(n))][y \leq S(n + 1)]$$

where

$$S(n) = \sum_{k \geq 1}^{n} k_{F}$$

and accordingly we now infer $(x,y,k,s,n \in \mathbb{N} \cup \{0\})$ [Remark 2.1.])

$$\zeta(x,y) = [x \leq y] - [x < y] \sum_{n \geq 0} [(x > S(n))][y \leq S(n + 1)].$$

Note. The author of [22–26, 40] consequently avoids the upside down notation. However she had used this notation then in her Rota and cobweb posets related disser-
tation that she had defended with distinction on 30 September 2008 [6]. The end of
comment.

No doubt the $\zeta$ function formulas – the former (Kwaśniewski) and the latter
(Krot) are valid for all natural numbers valued sequences $F$. Here is this other latter
form of Krot formula for $\zeta$ function (see: (7) in [22] or (1) in [24]).

Let $x = \langle s, t \rangle$ and $y = \langle u, v \rangle$ where $1 \leq s \leq F_t$, $1 \leq u \leq F_v$ while $t, v \in \mathbb{N}$. Then

$$\zeta(x, y) = \zeta((s, t), (u, v)) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t + k, v)$$

where here – recall ($a, b \in \mathbb{Z}$):

$$\delta(a, b) = \begin{cases} 
1 & \text{for } a = b, \\
0 & \text{otherwise}.
\end{cases}$$

In February 2009 – in the course of The Internet Gian Carlo Rota Polish Seminar
e-mail discussions with the present author – still another $\zeta$ – matrix formula was
postulated by Maciej Dzieniażčuk – in Knuth notation. See – below. We clame: all
are – up to the equivalence of description – the same. See then Comment 5.

Remark 4.1. Again on $\zeta$ formulas. Let us compare the above Krot formula for $\zeta$
with those by Kwaśniewski equivalent to the one from the Remark 3.1. $(x, y \in \mathbb{N})$ i.e.
with

$$\begin{align*}
\zeta(x, y) &= [x \leq y] - \sum_{s \geq 1} \sum_{k \geq 1} [x = k + sF][1 \leq y \leq sF + (s - 1)F - 1], \\
\zeta(x, y) &= [x \leq y] - \sum_{s \geq 1} [x > sF][1 \leq y \leq sF + (s - 1)F - 1],
\end{align*}$$

where, let us recall: $k, s \in \mathbb{N} \cup \{0\}$.

Let us rewrite the above Krot formula in Knuth notation keeping in mind the
conditions

$$1 \leq s \leq F_t, \ 1 \leq u \leq F_v, \ t, v \in \mathbb{N},$$

which should have been imposed altogether with:

$$\zeta(\langle s, t >, \langle u, v >) = [s = u][t = v] + [v > t].$$

The above formula with sine qua non conditions being implemented in there reads:

$$\zeta(\langle s, t >, \langle u, v >) = [s = u][t = v] + [v > t][1 \leq s \leq tF][1 \leq u \leq vF]$$

and so, if written with $\delta$’s it contains three subsequent summations as in the Kwaśniewski formula from 2003.

Acknowledgments

Thanks are expressed here to the now Student of Gdańsk University Maciej Dzie-
mańczuk for applying his skillful TeX-nology with respect most of my articles since
three years as well as for his general assistance and cooperation on KoDAGs investigation. Maciej Dziemiańczuk was not allowed to write his diploma with me being supervisor – while Maciej studied in the local Bialystok University where my professorship till 2009-09-30 comes from.

The author expresses his gratitude also Dr Ewa Krot-Sieniawska for her several years’ cooperation and vivid application of the alike material deserving Students’ admiration for her being such a comprehensible and reliable Teacher before she was fired by Bialystok University local authorities exactly on the day she had defended Rota and cobweb posets related dissertation with distinction.

References


Graded posets inverse zeta matrix formula


FORMUŁA NA MACIERZ MÔBIUSA DOWOLNEGO, CZĘŚCIOWO UPORZĄDKOWANEGO, ZBIORU Z GRADACJĄ I

Streszczenie

Część I jest zapiskiem historii badań zmierzających do otrzymania i udowodnienia formuły na macierz Môbiusa dowolnego częściowo uporządkowanego zbioru (“poset”) ze stopniowaniem (gradacją) o skończonej liczbie elementów minimalnych. Przypomina się te wcześniej otrzymaną postać macierzy incydencji dla wymienionych “posetów”.

W części I wprowadza się zarazem podstawowe pojęcia wsparte licznymi przykładami oraz odpowiednie techniki służące głównemu celowi określonego tytułem.